Study of Relative Equilibrium Positions of a Dynamically Symmetric Cubesat Nanosatellite under Aerodynamic and Gravitational Moments

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***Abstract*—The paper considers the angular motion of a dynamically symmetric CubeSat nanosatellite on a circular orbit under the aerodynamic and gravitational moments. Because of the rectangular shape of the CubeSat nanosatellite, the aerodynamic moment depends on two angles of orientation: angles of attack and proper rotation. The relative motion of CubeSat nanosatellites differs from the relative motion of axi-symmetric satellites. The formulas for calculating the relative equilibrium positions in the orbital reference frame for the dynamically symmetric CubeSat nanosatellite, when the center of mass is displaced from the geometric center along three axes, are obtained.**

Key words—CubeSat nanosatellite, aerodynamic moment, gravitational moment, angle of attack, angles of precession, angle of proper rotation

I. Introduction

Providing the required attitude of nanosatellites is of crucial importance because it influences the mission implementation. The required attitude of the nanosatellite can be provided by passive or combined stabilization systems. When developing such systems, it is necessary to take into account the nature of the uncontrolled motion of the nanosatellite relative to the center of mass under the influence of external moments. In order to study uncontrolled motion, an important task is to determine the equilibrium positions of the nanosatellite relative to the center of mass. Knowledge of the equilibrium positions and the nature of the nanosatellite motion in their vicinity allows using the moments of external forces for providing the necessary attitude. For example, works [1–3] show different realizations of the passive stabilization systems of CubeSat nanosatellite: one-axis aerodynamic, three-axis aerodynamic-gravitational, and three-axis gravitational-aerodynamic stabilizations.

There is a vast literature devoted to determination of the equilibrium positions relative to the mass center. For example, Sarychev studies dynamics of a satellite with a center of pressure displaced in three coordinates relative to the mass center and three different principal moments of inertia under the action of aerodynamic and gravitational moments. A symbol-numerical method for determining all equilibrium positions of a satellite in an orbital coordinate system is proposed [4–6]. In the works mentioned, "the effect of the atmosphere on the satellite is considered as a force of resistance applied at the center of pressure and directed against the velocity of the satellite's center of mass relative to the air" and this force does not depend on attitude of the satellite with respect to the incoming airflow, therefore it is constant, which is quite fair for the satellite shape close to spherical.

CubeSat nanosatellite has significant differences from other classes of satellites [1]. One of the most important distinctions is that it has a rectangular parallelepiped shape and, therefore, the drag coefficient of aerodynamic force depends on the nanosatellite attitude relative to the incoming airflow (angles of attack and proper rotation). Moreover, the maximum and minimum values of this coefficient differ significantly (for example, for CubeSat 3U nanosatellites they differ by more than 4 times depending on the orientation).

The equilibrium positions of the dynamically symmetric CubeSat nanosatellite when the center of mass is displaced from the geometric center along three axes under the influence of aerodynamic and gravitational moments are studied in the proposed work. The obtained formulas allow calculating the values of the angles of attack, precession and proper rotation, which correspond to the equilibrium positions, depending on the mass-inertial and geometric parameters of the nanosatellite, the orbital altitude, and the atmospheric density. We determine the conditions under which the number of relative equilibrium positions changes. It is shown that in the case that the gravitational moment dominates over the aerodynamic one, there are 16 equilibrium positions; when the aerodynamic moment dominates over the gravitational one, there are 8 equilibrium positions, and in the case when both moments have comparable values, there are 8, 12 or 16 equilibrium positions.

This paper is devoted to determining the equilibrium positions of the angular motion of the dynamically symmetric CubeSat nanosatellite in a circular orbit under the influence of aerodynamic and gravitational moments.

II. Mathematical Model

To describe motion of the nanosatellite around the center of mass, we used two reference frames: the trajectory reference frame *OXYZ* (which coincides with the orbital one in the case of circular orbit) and the body-fixed *Oxyz* reference frame (the axes are the main principal axes of inertia of the nanosatellite). To define the relation between the trajectory and the body-fixed reference frames, we used three Euler angles: α is the spatial angle of attack, ψ is the angle of precession, φ is the angle of proper rotation. The coefficients of the rotation matrix from the trajectory frame to the body-fixed frame are the following:

$$b_{11} = \cos\alpha\,,\ \ b_{12} = \sin\alpha\sin\psi\,,\ \ b_{13} = -\sin\alpha\cos\psi\,,$$

$$b_{21} = \sin\alpha\sin\varphi\,,\;\; b_{22} = \cos\varphi\cos\psi - \cos\alpha\sin\varphi\sin\psi\,,$$
$$b_{23} = \cos\varphi\sin\psi + \cos\alpha\sin\varphi\cos\psi\,,\;\; b_{31} = \sin\alpha\cos\varphi\,,$$
$$b_{32} = -\sin\varphi\cos\psi - \cos\alpha\cos\varphi\sin\psi\,,$$
$$b_{33} = -\sin\varphi\sin\psi + \cos\alpha\cos\varphi\cos\psi\,.$$

The gravitational moment projections on the body-fixed axes can be written as follows [7]:

$$M_{gx} = 3\frac{\mu}{R^3}(C-B)b_{23}b_{33},$$
$$M_{gy} = 3\frac{\mu}{R^3}(A-C)b_{13}b_{33}, \tag{1}$$
$$M_{gz} = 3\frac{\mu}{R^3}(B-A)b_{13}b_{23},$$

where A, B, C are the principal moments of inertia of the nanosatellite; μ is the gravitational constant of the Earth, R is the distance from the mass center to the center of attraction.

The aerodynamic moment for a CubeSat, which has the form of rectangular parallelepiped, can be represented in the body-fixed axes in the following form:

$$M_{ax} = -c_0 qS\cdot\tilde{S}(\alpha,\varphi)\cdot(\Delta y b_{31} - \Delta z b_{21}),$$
$$M_{ay} = -c_0 qS\cdot\tilde{S}(\alpha,\varphi)\cdot(\Delta z b_{11} - \Delta x b_{31}), \tag{2}$$
$$M_{az} = -c_0 qS\cdot\tilde{S}(\alpha,\varphi)\cdot(\Delta x b_{21} - \Delta y b_{11}),$$

where c_0 is the drag coefficient, which can take the values from 2 to 3, depending on the physical properties of the gas and the surface of the nanosatellite (for design studies, it is assumed to be 2.2); $q = \rho V^2/2$ is the velocity head; ρ is the current atmospheric density; V is the satellite flight velocity; $\tilde{S}(\alpha,\varphi) = |\cos\alpha| + k\sin\alpha\left(|\sin\varphi| + |\cos\varphi|\right)$ is the projection area of a CubeSat nanosatellite on a plane perpendicular to the velocity vector, divided by the characteristic area S [8]; $(\Delta x, \Delta y, \Delta z)$ are the coordinates of the center of pressure (geometric center) with respect to the center of mass; k is the ratio of the one lateral side surface area to the characteristic area.

The equations of the nanosatellite motion under the aerodynamic and gravitational moments in the circular orbit can be written in the following way [7]:

$$A\cdot\dot{\omega}_x + (C-B)\cdot\omega_y\cdot\omega_z = M_{gx} + M_{ax},$$
$$B\cdot\dot{\omega}_y + (A-C)\cdot\omega_z\cdot\omega_x = M_{gy} + M_{ay}, \tag{3}$$
$$C\cdot\dot{\omega}_z + (B-A)\cdot\omega_x\cdot\omega_y = M_{gz} + M_{az},$$

$$\omega_x = \dot{\psi}\cos\alpha + \dot{\varphi} + \omega_0 b_{12},$$
$$\omega_y = \dot{\psi}\sin\varphi\sin\alpha + \dot{\alpha}\cos\varphi + \omega_0 b_{22}, \tag{4}$$
$$\omega_z = \dot{\psi}\cos\varphi\sin\alpha - \dot{\alpha}\sin\varphi + \omega_0 b_{32}.$$

Here, ω_x, ω_y, ω_z are the projections of the absolute angular velocity vector on the body-fixed axes; ω_0 is the orbital angular velocity.

For the circular orbit, we can use the following formula:

$$\frac{\mu}{R^3} = \omega_0^2 \tag{5}$$

The equilibrium position is a position which the nanosatellite will be holding all the time in the case when at the initial moment of time, it was in that position and the velocities of all its points were zero [9]. According to the definition, the rate of change of the angles is zero ($\dot{\alpha} = 0$, $\dot{\psi} = 0$, $\dot{\varphi} = 0$) and (4) can be written as:

$$\omega_x = \omega_0\cdot b_{12},$$
$$\omega_y = \omega_0\cdot b_{22}, \tag{6}$$
$$\omega_z = \omega_0\cdot b_{32}.$$

Using (5) and (6) from (3), we obtain the system for determining the equilibrium positions:

$$\begin{gathered}
(C-B)\cdot\omega_0^2\left(b_{22}\cdot b_{32} - 3\cdot b_{23}\cdot b_{33}\right) + \\
+c_0 qS\left(|b_{11}| + k\left(|b_{21}| + |b_{31}|\right)\right)\left(\Delta y b_{31} - \Delta z b_{21}\right) = 0, \\
(A-C)\cdot\omega_0^2\left(b_{32}\cdot b_{12} - 3\cdot b_{33}\cdot b_{13}\right) + \\
+c_0 qS\left(|b_{11}| + k\left(|b_{21}| + |b_{31}|\right)\right)\left(\Delta z b_{11} - \Delta x b_{31}\right) = 0, \\
(B-A)\cdot\omega_0^2\left(b_{12}\cdot b_{22} - 3\cdot b_{13}\cdot b_{23}\right) + \\
+c_0 qS\left(|b_{11}| + k\left(|b_{21}| + |b_{31}|\right)\right)\left(\Delta x b_{21} - \Delta y b_{11}\right) = 0.
\end{gathered} \tag{7}$$

In this study we focus on the dynamically symmetric CubeSat nanosatellite when the mass center is displaced from the geometric center along three axes ($A \neq B = C$ and $\Delta x \neq 0, \Delta y \neq 0, \Delta z \neq 0$). In this case we can find the analytical solution from (7).

The final results are presented in Table 1, which shows a combination of angles of precession, proper rotation and attack, corresponding to the positions of angular equilibrium. We used the following notation:

$$w = k(|\Delta y| + |\Delta z|),\quad u = \left(\sqrt{w} + \sqrt{|\Delta x|}\right)^2,\quad v = \frac{\omega_0^2(B-A)}{c_0 qS},$$

$$\varphi_1 = \begin{cases} \arctan\dfrac{\Delta y}{\Delta z} + \pi, & \Delta z < 0, \\ \arctan\dfrac{\Delta y}{\Delta z}, & \Delta z > 0, \end{cases}\qquad \varphi_2 = \varphi_1 + \pi,$$

$$\alpha_i = \operatorname{arccot}\left(\frac{-r_i \pm \sqrt{r_i^2 + 4\Delta x q_i w}}{2q_i\sqrt{\Delta y^2 + \Delta z^2}}\right),$$

$r_i = q_i w + p_i v - \Delta x$, if i=1,2,5,6,9,10,13,14,

$$\alpha_i = \operatorname{arccot}\left(\frac{-r_i \pm \sqrt{r_i^2 - 4\Delta x q_i w}}{-2q_i\sqrt{\Delta y^2 + \Delta z^2}}\right) + \pi,$$

$r_i = q_i w + p_i v + \Delta x$, if i=3,4,7,8,11,12,15,16,

where " + " before the square root corresponds to the odd index α_i, respectively, and " – " corresponds to the even index; $p_i = -3$, if $i = 1,2,...,8$; $p_i = 1$, if $i = 9,10,...,16$; $q_i = 1$, if $i = 1,...,4;9,....,12$; $q_i = -1$, if $i = 5,...,8;13,...,16$.

We determined the conditions when the number of relative equilibrium positions varies:

- if $|v| < \frac{u}{3}$, there are 8 equilibrium positions due to the aerodynamic moment predominance over the gravitational one;
- if $\frac{u}{3} < |v| < u$, there are 12 equilibrium positions because both moments have comparable values;
- if $u < |v|$, there are 16 equilibrium positions due to gravitational moment predominance over the aerodynamic one.

Below are the results for the partial case where the center of mass is shifted only along the axis x (i.e. $A \neq B = C$, $\Delta x \neq 0$ and $\Delta y = \Delta z = 0$). In this case two equilibrium positions of the angle of attack $\alpha = 0$ and $\alpha = \pi$ (for any values of the angles of precession and proper rotation) exist for any gravitational and aerodynamic moments. When the aerodynamic moment influence decreases (i.e., when $v > |\Delta x|/3$ or $v > |\Delta x|$), additional equilibrium positions with respect to the angle of attack appear. The value of the angle of attack depends on the angle of proper rotation. The final results are presented in Table 2. We used the following notation:

$$\alpha_1 = \arctan\left(\frac{3v + \Delta x}{-\Delta x k\left(|\sin\varphi| + |\cos\varphi|\right)}\right),$$

$$\alpha_2 = \arctan\left(\frac{3v - \Delta x}{-\Delta x k\left(|\sin\varphi| + |\cos\varphi|\right)}\right) + \pi,$$

$$\alpha_3 = \arctan\left(\frac{v - \Delta x}{\Delta x k\left(|\sin\varphi| + |\cos\varphi|\right)}\right),$$

$$\alpha_4 = \arctan\left(\frac{v + \Delta x}{\Delta x k\left(|\sin\varphi| + |\cos\varphi|\right)}\right) + \pi.$$

TABLE 1. POSITIONS OF EQUILIBRIUM FOR THE CASE OF $A \neq B = C$ AND $\Delta x \neq 0, \Delta y \neq 0, \Delta z \neq 0$

			$\Delta x < 0$			$\Delta x > 0$		
			$\lvert v\rvert < \frac{u}{3}$	$\frac{u}{3} < \lvert v\rvert < u$	$u < \lvert v\rvert$	$\lvert v\rvert < \frac{u}{3}$	$\frac{u}{3} < \lvert v\rvert < u$	$u < \lvert v\rvert$
$\psi_1 = 0$ ($\psi_3 = \pi$)	φ_1	$v > 0$	α_3	$\alpha_1, \alpha_2, \alpha_3$	$\alpha_1, \alpha_2, \alpha_3$	α_1	$\alpha_1, \alpha_3, \alpha_4$	$\alpha_1, \alpha_3, \alpha_4$
		$v < 0$	α_3	α_3	α_3	α_1	α_1	α_1
	φ_2	$v > 0$	α_6	α_6	α_6	α_8	α_8	α_8
		$v < 0$	α_6	$\alpha_6, \alpha_7, \alpha_8$	$\alpha_6, \alpha_7, \alpha_8$	α_8	$\alpha_5, \alpha_6, \alpha_8$	$\alpha_5, \alpha_6, \alpha_8$
$\psi_2 = \frac{\pi}{2}$ ($\psi_4 = \frac{3\pi}{2}$)	φ_1	$v > 0$	α_{11}	α_{11}	α_{11}	α_9	α_9	α_9
		$v < 0$	α_{11}	α_{11}	$\alpha_9, \alpha_{10}, \alpha_{11}$	α_9	α_9	$\alpha_9, \alpha_{11}, \alpha_{12}$
	φ_2	$v > 0$	α_{14}	α_{14}	$\alpha_{14}, \alpha_{15}, \alpha_{16}$	α_{16}	α_{16}	$\alpha_{13}, \alpha_{14}, \alpha_{16}$
		$v < 0$	α_{14}	α_{14}	α_{14}	α_{16}	α_{16}	α_{16}
Equilibrium position number			**8**	**12**	**16**	**8**	**12**	**16**

TABLE 2. POSITIONS OF EQUILIBRIUM FOR THE CASE $A \neq B = C$ AND $\Delta x \neq 0$, $\Delta y = \Delta z = 0$

$\alpha = 0$ **or** $\alpha = \pi$ **for any values angles of precession and proper rotation**						
	($\Delta x > 0$ and $v > 0$) Or ($\Delta x < 0$ and $v < 0$)			($\Delta x < 0$ and $v > 0$) Or ($\Delta x > 0$ and $v < 0$)		
	$\lvert v\rvert < \frac{\lvert\Delta x\rvert}{3}$	$\frac{\lvert\Delta x\rvert}{3} < \lvert v\rvert < \lvert\Delta x\rvert$	$\lvert\Delta x\rvert < \lvert v\rvert$	$\lvert v\rvert < \frac{\lvert\Delta x\rvert}{3}$	$\frac{\lvert\Delta x\rvert}{3} < \lvert v\rvert < \lvert\Delta x\rvert$	$\lvert\Delta x\rvert < \lvert v\rvert$
$\psi_1 = 0$ ($\psi_3 = \pi$)	–	α_2	α_2	–	α_1	α_1
$\psi_2 = \frac{\pi}{2}$ ($\psi_4 = \frac{3\pi}{2}$)	–	–	α_3	–	–	α_4

III. Calculation of Equilibrium Positions for the SamSat-QB50

Using the formulas obtained in this work, we determined the equilibrium positions of the SamSat-QB50 nanosatellite. This nanosatellite was created at the Samara National Research University as part of an international university project QB50 and intended to study the Earth's thermosphere as part of nanosatellite constellation. [10]. We used the standard model of the atmospheric density (GOST 4401-81) [11].

The characteristics of the nanosatellite are the following:

$$m = 2.1\,kg,\ l = 0.32\ m,\ a = 0.1\,m,\ S = 0.01\,m^2,\ k = 2.5,$$
$$A = 0.0051\,kg\cdot m^2,\ B = C = 0.016\,kg\cdot m^2,\ c_0 = 2.2,$$
$$\Delta x = -0.061\,m,\quad \Delta y = -0.0013\ m,\quad \Delta z = -0.00053\ m\ .$$

For instance, we considered the altitude 560 km, where 16 relative equilibrium positions exist. The results are shown in Table 3.

Table 3. Equilibrium positions for the SamSat-QB50 nanosatellite at the altitude H=560 km

№	ψ	φ	α	№	ψ	φ	α
1	0	247.5°	0.3°	9	180°	247.5°	0.3°
2	0	247.5°	50.2°	10	180°	247.5°	50.2°
3	0	247.5°	179.8°	11	180°	247.5°	179.8°
4	0	67.5°	51.5°	12	180°	67.5°	51.5°
5	90°	247.5°	165.7°	13	270°	247.5°	165.7°
6	90°	67.5°	0.5°	14	270°	67.5°	0.5°
7	90°	67.5°	173°	15	270°	67.5°	173°
8	90°	67.5°	176.7°	16	270°	67.5°	176.7°

To sum up, in this work we have obtained formulas for determining the equilibrium positions of a dynamically symmetric CubeSat nanosatellite under the aerodynamic and gravitational moments for the circular orbits, when the CubeSat center-of-mass is displaced relative to the geometric center in three axes. It has been shown that there are no less than 8 positions of equilibrium. In the cases when the effect of the aerodynamic moment decreases, there are 12 or 16 positions of equilibrium. We have determined the conditions under which the number of relative equilibrium positions changes.

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